Finding a causal ordering via independent component analysis

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Abstract

We study the application of independent component analysis to discovery of a causal ordering between observed variables. Path analysis is a widely-used method for causal analysis. It is of confirmatory nature and can provide statistical tests for assumed causal relations based on comparison of the implied covariance matrix with a sample covariance. However, it is based on the assumption of normality and only uses the covariance structure, which is why it has several problems, for example, one cannot find the causal direction between two variables if only those two variables are observed because the two models to be compared are equivalent to each other. In previous work, we showed that use of nonnormality of observed variables can find the possible causal direction between two variables. In this article, we extend the method to more than two variables and develop a new statistical method for discovery of a causal ordering using nonnormality of observed variables.

Key words: Independent component analysis, nonnormality, independence, causal inference, nonexperimental data

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1 Introduction

An effective way to examine causality is to conduct an experiment with random assignment (Holland, 1986; Rubin, 1974). However, there are many situations that pose some difficulties to conduct experiments: One of the difficulties is that the direction of causality is often unknown. It is necessary to develop useful methods for finding a good inital model of causal orders between observed variables from nonexperimental data.

Path analysis was originated by the biologist S. Wright in 1920's and has been often applied to analyze causal relations of nonexperimental data in an empirical way. The path analysis is an extension of regression analysis where many endogenous and exogenous variables can be analyzed simultaneously. In 1970's, the path analysis was incorporated with factor analysis and latent variables were allowed in the model. The new framework is now called structural equation modeling (e.g., Bollen, 1989) and is a powerful tool of causal analysis.

However, the structural equation modeling (SEM) is of confirmatory nature and researchers have to model the true causal relationships based on background knowledge *before* collecting or analyzing data (e.g., Goldberger, 1972). It is difficult to model true causal relations in many cases, especially at the beginning of research. Lack of background knowledge often has the consequence that the causal direction is unknown.

Furthermore, SEM has some problems due to its restriction to normal distribution, for example: One cannot find the possible causal direction between two variables if only those two variables are observed because the two models with different direction are equivalent to each other.

A very simple illustration of the problem of finding the direction of causality is given by two regression models, called Model 1 and Model 2 here:

$$\text{Model 1:} \quad x_1 = b_{12}x_2 + \xi_1 \tag{1}$$
$$\text{Model 2:} \quad x_2 = b_{21}x_1 + \xi_2, \tag{2}$$

where the explanatory variable is assumed to be uncorrelated with the disturbance ξ_1 or ξ_2. We cannot say anything about which model is better from the two conventional regression analyses based on the two models above in the framework of SEM. Using the SEM terminology, the both models are saturated on the covariance matrix of $[x_1, x_2]$.

Kano and Shimizu (2003); Shimizu and Kano (2003b) showed that use of nonnormality of observed variables makes it possible to distinguish between

Model 1 and Model 2. In this paper, we shall extend their method to more than two variables and propose an algorithm to explore a causal ordering between observed variables from nonexperimental data.

2 Brief review of independent component analysis

Independent component analysis (ICA) is one of multivariate analysis techniques which aims at separating or recovering linearly-mixed unobserved multidimensional independent signals from the mixed observable variables. See e.g., Hyvärinen, Karhunen and Oja (2001), for a thorough description of ICA.

Let $\boldsymbol{x}$ be an observed m-vector. The ICA model for $\boldsymbol{x}$ is written as

$$\boldsymbol{x} = A\boldsymbol{s}, \tag{3}$$

where A is called a mixing matrix and $\boldsymbol{s}$ is an n-vector of unobserved variables or blind signals with zero mean and unit variance. Typically, the number of observed variables m is assumed to that of latent variables n. The main process of ICA is to estimate the mixing matrix.

Comon (1994) provided conditions for the model to be estimable for the typical case where $m \leq n$. The conditions include that the components of $\boldsymbol{s}$ are mutually independent and contain at most one normal component. ICA solves the estimation problem by maximizing independency among the components of $\boldsymbol{s}$. The independency is very often measured by nonnormality (see e.g., Hyvärinen and Kano, 2003). That is, the estimation is implemented by finding the demixing matrix W such that the components of $\hat{\boldsymbol{s}} = W\boldsymbol{x}$ have maximal nonnormality. A classical measure of nonnormality is kurtosis, defined as

$$\mathrm{kurt}(u) = E(u^4) - 3\{E(u^2)\}^2. \tag{4}$$

The kurtosis is zero for a normal variable and non-zero for most nonnormal variables. Comon (1994) proposed an estimation algorithm to maximize the sum of squared kurtosis of $\hat{\boldsymbol{s}}$, that is,

$$W = \arg\max_{W} \sum_{i=1}^{n} \mathrm{kurt}(\hat{s}_i)^2 = \arg\max_{W} \sum_{i=1}^{n} \mathrm{kurt}(\boldsymbol{w}_i^T\boldsymbol{x})^2, \tag{5}$$

where $\boldsymbol{w}_i^T$ denotes the i-th row of W. Here, the data is assumed to be sphered (whitened) (e.g., Hyvärinen, Karhunen and Oja, 2001) and W is constrained to be orthogonal.

Although the idea of ICA using kurtosis is simple, it can be very sensitive to outliers. Hyvärinen (1999) suggested a class of nonnormality measures

$$J(u) \propto [E(G(u)) - E(G(\nu))]^2, \tag{6}$$

where $G(\cdot)$ is a nonlinear and nonquadratic function and ν follows the normal distribution with zero mean and unit variance. More robust estimators are provided if the choice of G that does not grow too fast is made. For example, one can take $G(u) = \log\cosh(u)$. He further proposed a very efficient algorithm to estimate W maximizing (6), called FastICA (Hyvärinen, 1999; Hyvärinen and Oja, 1997).

In ICA as well as the traditional multivariate methods including factor analysis, the following ambiguities hold: i) one cannot determine the sign of s_i. one can multiply the independent component by -1 with giving no affect to the model in (3); ii) one cannot determine the order of the independent components. A permutation matrix P and its inverse can be substituted in the model to provide $\boldsymbol{x} = AP^{-1}P\boldsymbol{s}$. The element of $P\boldsymbol{s}$ are the original s_i, but in another order.

3 Finding a causal order between two variables

In this section, we shall explain how we can find a causal order between two variables using nonnormality.

3.1 Definition of a causal order

What is causality? Many philosophers and statisticians have tried to answer the quite difficult question and proposed various frameworks to find causal relations for a long time (Bollen, 1989; Bullock, Harlow and Mulaik, 1994; Granger, 1969; Holland, 1986; Hume, 1740; Mill, 1843; Mulaik and James, 1995; Pearl, 2000; Rubin, 1974; Suppes, 1970).

In this article, we say that causality (a causal order) from a random variable x_1 to a random variable x_2, which we denote by $x_1 \to x_2$, is confirmed if an equation:

$$x_2 = f(x_1, \xi_2) \tag{7}$$

holds where ξ_2 is a disturbance variable which is independently distributed

from the explanatory variable x_1.[3] The ξ_2 is a function of many variables $z_1, z_2, \cdots, z_q$ that have small and not very important influences on x_2 or that may not be noticed by researcher, as well as an error variable e_2. That is, $\xi_2 = g(z_1, z_2, \cdots, z_q, e_2)$ (e.g., Bollen, 1989).

For simplicity, let us assume that $f(x_1, \xi_2) = b_{21}x_1 + \xi_2$, is a simple linear function of x_1 and ξ_2. Then we obtain a simple regression analysis model:

$$x_2 = b_{21}x_1 + \xi_2, \tag{8}$$

where x_1 and ξ_2 are independent from each other. Now we can reformulate the causal order of x_1 to x_2: a nonzero constant b_{21} exists so that (8) holds. Note that independence between an explanatory variable x_1 and a disturbance variable ξ_2, not only their uncorrelatedness, is assumed here.[4]

The two concepts, independence and uncorrelatedness are very different. The independence between s_1 and s_2 is equivalent to

$$E[h_1(s_1)h_2(s_2)] - E[h_1(s_1)]E[h_2(s_2)] = 0. \tag{9}$$

for any two functions h_1 and h_2. Uncorrelatedness is a much weaker condition than independence. Two random variables s_1 and s_2 are said to be uncorrelated if their covariance is zero,

$$E(s_1s_2) - E(s_1)E(s_2) = 0. \tag{10}$$

If those two variables are independent, they are uncorrelated, which follows directly from (9) taking $h_1(s_1) = s_1$ and $h_2(s_2) = s_2$. However, uncorrelatedness does not mean independence (see, e.g., Hyvärinen and Oja, 2000).

A dependency between x_1 and ξ_2 would imply the existence of one (or more) unobserved confounding variables between x_1 and x_2 (Bollen, 1989; Kano and Shimizu, 2003). It is known that regression-based causal analysis may be completely distorted if there are unobserved confounding variables. If x_1 and ξ_2

[3] Rigorously speaking, we need to examine if equation (7) holds for each unit in a population U to confirm causation from x_1 to x_2 in U because we have to distinguish between interpersonal change (causation) and individual difference (association) (see, e.g., Holland, 1986, for causation and association). However, it is rarely possible to examine it from nonexperimental data since the data is usually one-time-point data. In this article, we assume that interpersonal change can be approximated by individual difference in our data sets, which is usually assumed in causal analysis based on nonexperimental data.

[4] The condition is related to pseudo-isolation in Bollen (1989). However, he required only uncorrelatedness, not independence.

are independent, it implies that no unobserved confounding variable exists (Kano and Shimizu, 2003). However, if they are merely uncorrelated, it does not ensure anything about the existence of confounding variables. Let z be an unobserved confounding variable, and let us assume that

$$x_2 = b_{21}x_1 + \gamma_{23}z + \xi_2 \tag{11}$$
$$x_1 = \gamma_{13}z + \xi_1. \tag{12}$$

We then have

$$\mathrm{Cov}(x_1, x_2) = b_{21}\mathrm{Var}(x_1) + \gamma_{23}\gamma_{13}\mathrm{Var}(z). \tag{13}$$

Depending on the particular values of γ_{23} and γ_{13}, there could be nonzero covariance between x_1 and x_2 even if b_{21}=0, and one could make an interpretation that a causal order from x_2 to x_1 or its opposite exists; on the other hand, there could be zero covariance between x_1 and x_2 even if b_{21} is large enough. Thus, independence and nonnormality are key assumptions in our settings.

3.2 Finding a causal order between two variables

Let x_{1j} and x_{2j} $(j = 1, \ldots, N)$ be observations on random variables x_1 and x_2 with zero mean. Denote $\overline{x_i^2} = \frac{1}{N}\sum_{j=1}^{N} x_{ij}^2$ $(i = 1, 2)$ and $\overline{x_1 x_2} = \frac{1}{N}\sum_{j=1}^{N} x_{1j}x_{2j}$. We shall use similar notation in subsequent derivations without explicit definitions.

The second-order moment structure of Model 1 is obviously given as

$$E\begin{bmatrix} \overline{x_1^2} \\ \overline{x_1 x_2} \\ \overline{x_2^2} \end{bmatrix} = \begin{bmatrix} b_{12}^2 E(x_2^2) + E(\xi_1^2) \\ b_{12}E(x_2^2) \\ E(x_2^2) \end{bmatrix} \quad \text{which we denote by} \quad E[\boldsymbol{m}_2] = \boldsymbol{\sigma}_2(\boldsymbol{\tau}_2),$$

where $\boldsymbol{\tau}_2 = [E(x_2^2),\ E(\xi_1^2),\ b_{12}]^T$. The number of sample moments to be used and the number of parameters are both three and thus, the Models 1 and 2 are saturated and equivalent to each other as far as covariances alone are concerned. Both models receive a perfect fit to the sample covariance matrix.

Shimizu and Kano (2003b) assumed that $[x_1, x_2]$ is nonnormally distributed and utilized higher-order moments to distinguish between Model 1 and Model 2. They further assumed that explanatory and disturbance variables, x_2 and ξ_1, x_1 and ξ_2, are independently distributed.

Consider using fourth-order moments. The expectations of the fourth-order moments can be expressed in a similar manner as

$$E\begin{bmatrix} \overline{x_1^4} \\ \overline{x_1^3 x_2} \\ \overline{x_1^2 x_2^2} \\ \overline{x_1 x_2^3} \\ \overline{x_2^4} \end{bmatrix} = \begin{bmatrix} b_{12}^4 E(x_2^4) + 6b_{12}^2 E(x_2^2)E(\xi_1^2) + E(\xi_1^4) \\ b_{12}^3 E(x_2^4) + 3b_{12} E(x_2^2)E(\xi_1^2) \\ b_{12}^2 E(x_2^4) + E(x_2^2)E(\xi_1^2) \\ b_{12} E(x_2^4) \\ E(x_2^4) \end{bmatrix}$$

which we denote by $E[\boldsymbol{m}_4] = \boldsymbol{\sigma}_4(\boldsymbol{\tau}_4)$

for Model 1, where $\boldsymbol{\tau}_4 = [\boldsymbol{\tau}_2^T,\ E(x_2^4),\ E(\xi_1^4)]^T$.

In Model 1, we have three second-order moments and five fourth-order moments, whereas there are five parameters. The number of parameters is smaller than the number of moments used. Thus, if we define a measure of model fit by a weighted distance between the observed moments and the moments implied by the model as

$$T = N \left(\begin{bmatrix} \boldsymbol{m}_2 \\ \boldsymbol{m}_4 \end{bmatrix} - \begin{bmatrix} \boldsymbol{\sigma}_2(\widehat{\boldsymbol{\tau}}_2) \\ \boldsymbol{\sigma}_4(\widehat{\boldsymbol{\tau}}_4) \end{bmatrix} \right)^T \hat{M} \left(\begin{bmatrix} \boldsymbol{m}_2 \\ \boldsymbol{m}_4 \end{bmatrix} - \begin{bmatrix} \boldsymbol{\sigma}_2(\widehat{\boldsymbol{\tau}}_2) \\ \boldsymbol{\sigma}_4(\widehat{\boldsymbol{\tau}}_4) \end{bmatrix} \right) \tag{14}$$

with appropriate estimators $\widehat{\boldsymbol{\tau}}_i$ and a correctly chosen weight matrix $\hat{M}$, then T represents distance between data and the model employed and will be asymptotically distributed according to the chi-square distribution with df= 3 degrees of freedom. See Section 5 for some details. We can thus evaluate a fit of Model 1 using the statistic T. The same argument holds for Model 2, and we can confirm that Models 1 and 2 are not equivalent to each other in general, that is, the independence assumption between explanatory and disturbance variables is better fitted to one model than the other.

4 Finding a causal ordering (causal orders) between more than two variables based on ICA

In this section, we propose a new method of finding causal orders that generalizes our previous work, reviewed in the preceding section, to more than two variables.

4.1 Definition of a causal ordering

We say that observed variables x_i have a causal ordering if they can be ordered so that each variable is a function of the preceding variables plus an independent disturbance variable ξ_i. Let us denote this ordering by $i(1), \ldots, i(n)$.

In other words, we say that random variables, $x_1, x_2, \cdots, x_n$, have a causal ordering, $x_{i(1)} \rightarrow x_{i(2)} \rightarrow \cdots \rightarrow x_{i(n)}$, if nonzero coefficients $\beta_{i(j),i(k)}$ $(j = 1, 2, \cdots, n, k < j)$ exist so that the equations:

$$x_{i(j)} = \sum_{k<j} \beta_{i(j),i(k)} x_{i(k)} + \xi_{i(j)}, \tag{15}$$

hold where $\xi_{i(j)}$ is a disturbance variable and is independently distributed from $x_{i(k)}$ and from $\xi_{i(k)}$ for all $k < j$.

4.2 Definition of data model

Our definition of causality in (15) can also be interpreted as a data model. In the following, we actually assume that the data follows such a model so that the causal ordering is possible to find. Thus, we assume the following data model:

$$x_{i(j)} = \sum_{k<j} b_{i(j),i(k)} x_{i(k)} + \xi_{i(j)}. \tag{16}$$

We also assume that the disturbance variables $\xi_{i(j)}$ are nonnormal, and mutually independent. This implies that $\xi_{i(j)}$ is independent from $\xi_{i(k)}$ for all $k < j$.

To investigate the causal structure of the x_i, we would like to find the correct ordering $i(j)$. Thus, the problem is finding the permutation of the observed variables that reflects the causal structure of the data. In what follows, we will show how such an ordering can be identified.

4.3 Estimation of model

Let us normalize the equation (16) so that the disturbance variables ξ_i have unit norm. Denoting

$$w_{i(j),i(j)} = 1/\sqrt{\mathrm{var}(\xi_{i(j)})} \tag{17}$$

$$w_{i(j),i(k)} = -b_{i(j),i(k)}/\sqrt{\mathrm{var}(\xi_{i(j)})} \quad \text{for } k \neq j, \tag{18}$$

the equation (16) can be expressed as:

$$w_{i(j),i(j)} x_{i(j)} = \sum_{k<j} -w_{i(j),i(k)} x_{i(k)} + \xi^*_{i(j)}, \tag{19}$$

where $\xi^*_{i(j)}$ are the disturbance variables standardized to have unit variance.

Let us denote by $\tilde{\boldsymbol{x}}$ the vector where the observed variables are ordered according to $i(j)$. In matrix form, equation (16) can be expressed as

$$\tilde{\boldsymbol{x}} = B\tilde{\boldsymbol{x}} + \tilde{\boldsymbol{\xi}} \tag{20}$$

where the matrix B is lower triangular. Using W, this becomes

$$\mathrm{diag}(W)\tilde{\boldsymbol{x}} = -\mathrm{offdiag}(W)\tilde{\boldsymbol{x}} + \tilde{\boldsymbol{\xi}}^* \quad \text{or equivalently} \quad W\tilde{\boldsymbol{x}} = \tilde{\boldsymbol{\xi}}^* \tag{21}$$

where W is still lower triangular, for the correct permutation of the observed variables. This corresponds to the correct permutation of the columns of W. From the theory of ICA, we know that this W can be estimated up to a permutation of its rows, using standard ICA methods.

Now we can use the following theorem:

Theorem 1 *If W is lower triangular and all the elements w_{ij} are nonzero for $i \geq j$, no other permutation of rows and columns is lower triangular*

Proof First, note that any joint permutation of rows and columns can be performed by first permuting the rows and then the columns. This is because the permutations of rows or columns can be expressed by left and right multiplication by permutation matrices, respectively, and any product of multiple permutations therefore reduces to a multiplication by two permutation matrices, one from the right and one from the left, and either of the multiplications can be done first. Assume a permutation of rows has been done, and denote this new matrix by $W^\dagger$. Assume that the first row in $W^\dagger$ is not the same as the first row in W. Then, at least two elements on the first row of $W^\dagger$ are nonzero. Now, any permutation of columns cannot change the number of nonzero elements on the first row. Thus, a combination of row and column permutation that is lower-triangular must be such that the first row of the row-permuted

matrix $W^{\dagger}$ is equal to the first row of W. Also, the column-permutation cannot move the first column in order to preserve lower-triangularity. Thus, we have proven that the first row must remain the first row, and the first column must remain the first column. The same proof can be applied on every row and column in succession, which proves the theorem.

Therefore, if the $b_{i(j),i(k)}$ are not zeros, the permutation to make

$$W = \mathrm{Var}(\boldsymbol{\xi})^{-1/2}(I_n - B) \tag{22}$$

to lower triangular is unique. The I_n denotes an n-dimensional identity matrix. Then the causal ordering between x_i is uniquely determined taking the $b_{i(j),i(k)}$ as the $\beta_{i(j),i(k)}$ in (15).

We propose a simple algorithm for finding the optimal permutations in W.

(1) Find the $m(m-1)/2$-th least element in the absolute values of W and denote the value by c.
(2) Sort the rows of W in ascending order of the number of elements whose absolute values are greater than c.
(3) Sort the columns of W in descending order of the number of elements whose absolute values are greater than c.

The validity of our algorithm can be proven as follows. Assume W is a row/column permuted version of a lower-triangular matrix. Then, we have $c = 0$. Since any permutation of the columns cannot change the number of non-zero elements in each row, then the number of elements whose absolute values are greater than c in the i-th row will be equal to the index of the row in the original lower-triangular matrix. Thus, step (2) will find the correct permutation of rows. Likewise, step (3) will find the correct permutation of columns.

If we have correctly permuted W, the disturbance standard deviation $\sqrt{\mathrm{var}(\xi_{i(j)})}$ can be estimated by $1/w_{i(j),i(j)}$ from (17). [5] Then we obtain the estimate of B by

$$\hat{B} = I_n - \mathrm{diag}(\hat{W})^{-1}\hat{W}. \tag{23}$$

Thus, the model (20) can be estimated by

(1) estimating an initial W by ICA,

[5] ICA has the sign ambiguity. The estimated w_{ii} could have a negative sign. Then we multiply the i-th row vector $\boldsymbol{w}_i^T$ by -1 so that w_{ii} has a positive sign.

(2) finding a combination of permutations of the rows and the columns of $\tilde{W}$ so that $\tilde{W}$ becomes as close to lower triangular as possible, using the algorithm above,
(3) estimating B by $I_n - \mathrm{diag}(\tilde{W})^{-1}\tilde{W}$.

The $\tilde{W}$ denotes a correctly permuted version of W. It should be noted that the correct causal ordering is given by the permutation of the rows found by our method. The correct permutation of columns and the value of B are additional information that are not always necessary.

4.4 Example

Now we shall show the models 1 and 2 can be expressed in this framework. In Model 1, the causal order of observed variables is $(i(1), i(2)) = (2, 1)$. Model 1 can be rewritten as:

$$\text{Model 1:} \quad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & b_{12} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} \tag{24}$$

$$\Leftrightarrow \begin{bmatrix} x_2 \\ x_1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ b_{12} & 0 \end{bmatrix} \begin{bmatrix} x_2 \\ x_1 \end{bmatrix} + \begin{bmatrix} \xi_2 \\ \xi_1 \end{bmatrix}. \tag{25}$$

Here $\tilde{\boldsymbol{x}}$ and B are

$$\tilde{\boldsymbol{x}} = \begin{bmatrix} x_2 \\ x_1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ b_{12} & 0 \end{bmatrix}. \tag{26}$$

One can see that the B is lower triangular when the observed variables are ordered according to $i(j)$. Also in Model 2, one can see the lower triangularity of B in the same manner.

4.5 Alternative approach

Above, we said that the causal ordering between observed variables is uniquely determined if all the $b_{i(j),i(k)}$ are not zeros, which is a necessary but not the sufficient condition. There is another possibility where the causal ordering is unique. Let A be the inverse of W. Note that A is also lower triangular. The a_{ij}/a_{jj} represents the total effect of x_j to x_i, whereas the b_{ij} the direct effect of

x_j to x_i (see, Bollen, 1989, for total effect and direct effect). Then the model (20) can be rewritten as

$$\tilde{\boldsymbol{x}} = A\tilde{\boldsymbol{\xi}}^*, \tag{27}$$

which is the ICA model in (3) and the A is estimable up to a permutation of its columns, using standard ICA methods. We can find the optimal permutations in A in the same manner as finding those in W. Now, the causal ordering between x_i is uniquely determined if $a_{i(j),i(k)}$ are not zeros (Theorem 1).

Taking $a_{i(j),i(k)}/a_{i(k),i(k)}$ as $\beta_{i(j),i(k)}$ in (15), the link between the lower triangularity of A and the causal ordering can be seen as follows. For the lower triangular mixing matrix, $x_{i(1)}$ is essentially equal to $\xi^*_{i(1)}$, up to a multiplicative constant, $a_{i(1),i(1)}$. On the other hand, $x_{i(2)}$ is a function of $\xi^*_{i(1)}$ and $\xi^*_{i(2)}$, $a_{i(2),i(1)}\xi^*_{i(1)} + a_{i(2),i(2)}\xi^*_{i(2)}$. Thus, $x_{i(2)}$ is a function of $x_{i(1)}$ and a new independent variable, $\xi^*_{i(2)}$, that is, $(a_{i(2),i(1)}/a_{i(1),i(1)})x_{i(1)} + a_{i(2),i(2)}\xi^*_{i(2)}$. This indicates that $x_{i(1)}$ may cause $x_{i(2)}$, but $x_{i(2)}$ cannot cause $x_{i(1)}$. Continuing the same logic, we see that $x_{i(1)}$ can cause $x_{i(3)}$ and $x_{i(2)}$ can cause $x_{i(3)}$, but $x_{i(3)}$ cannot cause either $x_{i(1)}$ or $x_{i(2)}$ because $x_{i(3)}$ is simply a function of $x_{i(1)}$ and $x_{i(2)}$. In general, $x_{i(j)}$ is a function of $x_{i(1)}, \cdots, x_{i(j-1)}$ and $\xi^*_{i(j)}$, which establishes the direction of possible causality.

The two methods: i) W-based method; ii) A-based method compensate each other. For example, let us assume that

$$\begin{bmatrix} x_{i(1)} \\ x_{i(2)} \\ x_{i(3)} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ b_{21} & 0 & 0 \\ 0 & b_{32} & 0 \end{bmatrix} \begin{bmatrix} x_{i(1)} \\ x_{i(2)} \\ x_{i(3)} \end{bmatrix} + \begin{bmatrix} \xi_{i(1)} \\ \xi_{i(2)} \\ \xi_{i(3)} \end{bmatrix}, \tag{28}$$

where b_{21} and b_{32} are not zeros. The b_{31} is zero and the causal ordering may not be unique if the W-based method is applied. However, let us rewrite (28) as

$$\begin{aligned} \begin{bmatrix} x_{i(1)} \\ x_{i(2)} \\ x_{i(3)} \end{bmatrix} &= \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ b_{21} & 0 & 0 \\ 0 & b_{32} & 0 \end{bmatrix} \right)^{-1} \begin{bmatrix} \mathrm{Var}(\xi_{i(1)}) & 0 & 0 \\ 0 & \mathrm{Var}(\xi_{i(2)}) & 0 \\ 0 & 0 & \mathrm{Var}(\xi_{i(3)}) \end{bmatrix}^{1/2} \begin{bmatrix} \xi^*_{i(1)} \\ \xi^*_{i(2)} \\ \xi^*_{i(3)} \end{bmatrix} \\ &= \begin{bmatrix} \mathrm{Var}(\xi_{i(1)})^{1/2} & 0 & 0 \\ b_{21}\mathrm{Var}(\xi_{i(1)})^{1/2} & \mathrm{Var}(\xi_{i(2)})^{1/2} & 0 \\ b_{32}b_{21}\mathrm{Var}(\xi_{i(1)})^{1/2} & b_{32}\mathrm{Var}(\xi_{i(2)})^{1/2} & \mathrm{Var}(\xi_{i(3)})^{1/2} \end{bmatrix} \begin{bmatrix} \xi^*_{i(1)} \\ \xi^*_{i(2)} \\ \xi^*_{i(3)} \end{bmatrix}. \end{aligned} \tag{29}$$

Then the causal ordering can be recovered by the A-based method.

Another simple example is:

$$\begin{bmatrix} x_{i(1)} \\ x_{i(2)} \\ x_{i(3)} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ b_{21} & 0 & 0 \\ b_{31} & b_{32} & 0 \end{bmatrix} \begin{bmatrix} x_{i(1)} \\ x_{i(2)} \\ x_{i(3)} \end{bmatrix} + \begin{bmatrix} \xi_{i(1)} \\ \xi_{i(2)} \\ \xi_{i(3)} \end{bmatrix}, \tag{30}$$

where b_{21}, b_{31}, b_{32} are not zeros and $b_{31} + b_{32}b_{21}$ is zero, for example, $b_{21} = 0.3, b_{31} = 0.6, b_{32} = -0.2$. Then all the direct effect of $x_{i(k)}$ to $x_{i(j)}$, $b_{i(j),i(k)}$ $(k < j)$, are not zeros and the causal ordering can be recovered by the W-based method. However, the A-based method fails to recover the causal ordering because the total effect of $x_{i(1)}$ to $x_{i(3)}$, $b_{31} + b_{32}b_{21}$, is zero:

$$\begin{bmatrix} x_{i(1)} \\ x_{i(2)} \\ x_{i(3)} \end{bmatrix} = \begin{bmatrix} \mathrm{Var}(\xi_{i(1)})^{1/2} & 0 & 0 \\ b_{21}\mathrm{Var}(\xi_{i(1)})^{1/2} & \mathrm{Var}(\xi_{i(2)})^{1/2} & 0 \\ (b_{31} + b_{32}b_{21})\mathrm{Var}(\xi_{i(1)})^{1/2} \ (=0) & b_{32}\mathrm{Var}(\xi_{i(2)})^{1/2} & \mathrm{Var}(\xi_{i(3)})^{1/2} \end{bmatrix} \begin{bmatrix} \xi^*_{i(1)} \\ \xi^*_{i(2)} \\ \xi^*_{i(3)} \end{bmatrix}.$$

Thus both W-based and A-based methods are useful for finding a causal ordering between observed variables. In the latter part of this article, we report the simulation experiment and real example on the W-based method for saving space.

5 Examination of independence

In our setting, the independence assumption between explanatory and disturbance variables is crucial. We propose a test statistic to examine the independence assumption statistically.

Let N be a sample size and define V as

$$V = \lim_{N\to\infty} N \times \mathrm{Var}[\boldsymbol{m}_2^T,\ \boldsymbol{m}_4^T]^T. \tag{31}$$

Letting $\boldsymbol{\tau}$ be a vector that contains the model parameters and $\boldsymbol{m}_2$ and $\boldsymbol{m}_4$ be the vectorized second- and fourth-order moments after removing the redundant elements and $\boldsymbol{\sigma}_2(\boldsymbol{\tau}) = E(\boldsymbol{m}_2)$, $\boldsymbol{\sigma}_4(\boldsymbol{\tau}) = E(\boldsymbol{m}_4)$, the test statistic T to examine the model assumption is defined as

$$T = N \left(\begin{bmatrix} \boldsymbol{m}_2 \\ \boldsymbol{m}_4 \end{bmatrix} - \begin{bmatrix} \boldsymbol{\sigma}_2(\hat{\boldsymbol{\tau}}) \\ \boldsymbol{\sigma}_4(\hat{\boldsymbol{\tau}}) \end{bmatrix} \right)^T \hat{M} \left(\begin{bmatrix} \boldsymbol{m}_2 \\ \boldsymbol{m}_4 \end{bmatrix} - \begin{bmatrix} \boldsymbol{\sigma}_2(\hat{\boldsymbol{\tau}}) \\ \boldsymbol{\sigma}_4(\hat{\boldsymbol{\tau}}) \end{bmatrix} \right), \tag{32}$$

with

$$\hat{M} = \hat{V}^{-1} - \hat{V}^{-1}\hat{J}(\hat{J}^T\hat{V}^{-1}\hat{J})^{-1}\hat{J}^T\hat{V}^{-1}, \tag{33}$$

where

$$\hat{J} = \left. \frac{\partial [\boldsymbol{\sigma}_2(\boldsymbol{\tau})^T, \boldsymbol{\sigma}_4(\boldsymbol{\tau})^T]^T}{\partial \boldsymbol{\tau}^T} \right|_{\boldsymbol{\tau}=\hat{\boldsymbol{\tau}}}. \tag{34}$$

The statistic T approximates to a chi-square variate with degrees $\mathrm{tr}[VM]$ of freedom where N is large enough (e.g., Shimizu and Kano, 2003a). The required assumption for this is that $\hat{\boldsymbol{\tau}}$ is a $\sqrt{N}$-consistent estimator. No asymptotic normality is needed. See Browne (1984) for details.

6 Simulation experiment

We conducted a small simulation experiment to study the performance of the method described above. The simulation consisted of 100 causal ordering recovery trials. In each trial, we generated twenty-dimensional random variables $\tilde{\boldsymbol{\xi}}^*$ of sample size $N = 50000$ as standardized disturbance variables where their components are independently distributed according to the t distribution with parameters yielding kurtoses from 6 to 2. The disturbance variables were standardized to have zero mean and unit variance. A random lower-triangular matrix B where the element b_{ij} $(i > j)$ was distributed according to the uniform distribution $U(0.2, 1)$ and multiplied by -1 with probability 50% was created. A random diagonal matrix D was created in the same manner. Then a random mixing matrix $A = (I_{20} - B)D$ was computed.

The standardized disturbance variables were linearly mixed by A after both rows and columns were permuted randomly.

We employed FastICA [6] as an ICA method and took $\log\cosh(u)$ as $G(u)$ in (6), where the symmetric orthogonalization was applied (Hyvärinen, 1999; Hyvärinen and Oja, 1997).

[6] The MATLAB package is available at http://www.cis.hut.fi/projects/ica/fastica/.

The W-based method developed above was then applied on the data. The performance of our method was evaluated as follows. We computed how many diagonal elements in the matrix $\hat{W}A$ had an absolute value that was larger than 0.99, which provided a measure of how many causal orders had been recovered. Though in the ideal case where $\hat{W}A$ is a signed identity matrix, in practice, some errors occurs in the estimation of the W. Thus we took the value 0.99 as the threshold. The error in the estimation of the B was assessed using the root mean square error:

$$\sqrt{\mathrm{tr}[(\hat{B}-B)(\hat{B}-B)^T]/n^2}. \tag{35}$$

The W-based method recovered 100% of the causal orders. The root mean square error (35) was 0.02.

7 Real data example

Questionnaire data about computer literacy learning were analyzed as an example to illustrate the effectiveness of our method described here. The survey was conducted at Osaka University in 2002 (c.f., Torii, 2004) to study computer anxiety. The sample size was 272. Observed variables were standardized so that all the variables have zero mean and unit variance.

We explored a causal ordering between x_1, x_2 and x_3 using the W-based method proposed in Section 4. We employed FastICA, where $\log\cosh(u)$ is taken as $G(u)$ in (6) and the symmetric orthogonalization was applied. The labels of the observed variables x_1, x_2 and x_3 are shown in Table 1.

Table 1
Variable labels

x_1: Subjective evaluation on your proficiency at the beginning of the class
x_2: Subjective evaluation on your operation anxiety
x_3: Subjective evaluation on your technology anxiety, or negative belief toward the computer

The estimated W by FastICA was

$$\begin{bmatrix} 0.99 & -0.01 & -0.10 \\ 0.61 & 1.21 & -0.49 \\ -0.31 & -0.27 & -0.91 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \xi_1^* \\ \xi_2^* \\ \xi_3^* \end{bmatrix}, \tag{36}$$

and the permuted W so that it becomes as lower triangular as possible was

$$\begin{bmatrix} 0.99 & -0.10 & -0.01 \\ -0.31 & -0.91 & -0.27 \\ 0.61 & -0.49 & 1.21 \end{bmatrix} \begin{bmatrix} x_1 \\ x_3 \\ x_2 \end{bmatrix} = \begin{bmatrix} \xi_1^* \\ \xi_3^* \\ \xi_2^* \end{bmatrix}, \tag{37}$$

where the second and the third rows and the second and the third columns were permuted, respectively. The independence assumption between standardized disturbance variables, $\xi_1^*, \xi_2^*, \xi_3^*$ was not rejected (T in (32) was 4.72 with p value of 0.86), which implies that no unobserved confounding variables existed. The result implies the causal ordering, $x_1 \to x_3 \to x_2$, that is, proficiency at the beginning of the class $\to$ technology anxiety $\to$ operation anxiety, which would be reasonable to the computer anxiety theory in computer literacy learning.

8 Discussion

We developed a new statistical method for discovering a possible causal ordering using nonnormality of observed variables. Whereas there are some approaches on causal analysis such as SEM, our approach based on ICA is totally different from them. SEM cannot find the direction of causality in many cases without much background knowledge because the normal assumption on SEM limits its applicability. We provided a partial solution to the problem utilizing nonnormality of observed variables.

There are some drawbacks of our model. When the distribution is close to the normal distribution, our method is unstable. Linearity assumption is rather restrictive. The complete recursiveness assumption is also restrictive.

Researchers should and can make further confirmatory causal inferences including experimental and longitudinal studies based on the result of our exploratory causal inference method. The method developed here would be helpful to construct a good initial model.

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